

# GLOBAL STRONG SOLUTIONS TO THE PLANAR COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH LARGE INITIAL DATA AND VACUUM

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**ABSTRACT.** This paper considers the initial boundary problem to the planar compressible magnetohydrodynamic equations with large initial data and vacuum. The global existence and uniqueness of large strong solutions are established when the heat conductivity coefficient  $\kappa(\theta)$  satisfies, for any  $q > 0$  and  $C > 0$ , that

$$C^{-1}(1 + \theta^q) \leq \kappa(\theta) \leq C(1 + \theta^q).$$

## 1. INTRODUCTION

Magnetohydrodynamics (MHD) studies the dynamics of conducting fluids in an magnetic field. The MHD finds its way in a very wide range of physical objects, from liquid metals to cosmic plasmas, for example, see [3, 23, 30, 33, 37]. The governing equations of compressible planar magnetohydrodynamic flows, which implies that the flows are uniform in the transverse directions, take the following form:

$$\rho_t + (\rho u)_x = 0, \quad (1.1)$$

$$(\rho u)_t + \left( \rho u^2 + P + \frac{1}{2} |\mathbf{b}|^2 \right)_x = (\lambda u_x)_x, \quad (1.2)$$

$$(\rho \mathbf{w})_t + (\rho u \mathbf{w} - \mathbf{b})_x = (\mu \mathbf{w}_x)_x, \quad (1.3)$$

$$\mathbf{b}_t + (u \mathbf{b} - \mathbf{w})_x = (\nu \mathbf{b}_x)_x, \quad (1.4)$$

$$(\rho e)_t + (\rho u e)_x - (\kappa e_x)_x = \lambda u_x^2 + \mu |\mathbf{w}_x|^2 + \nu |\mathbf{b}_x|^2 - P u_x, \quad (1.5)$$

where the unknowns  $\rho \geq 0$  denotes the density of the flow,  $u \in \mathbb{R}$  the longitudinal velocity,  $\mathbf{w} \in \mathbb{R}^2$  the transverse velocity,  $\mathbf{b} \in \mathbb{R}^2$  the transverse magnetic field, and  $e$  the internal energy, respectively. Both the pressure  $P$  and the internal energy  $e$  are generally related to the density and temperature of the flow according to the equations of state:  $P = P(\rho, \theta)$  and  $e = e(\rho, \theta)$ . The parameters  $\lambda = \lambda(\rho, \theta)$  and  $\mu = \mu(\rho, \theta)$  denote the bulk and the shear viscosity coefficients, respectively;  $\nu = \nu(\rho, \theta)$  is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, and  $\kappa = \kappa(\rho, \theta)$  is the heat conductivity.

The system (1.1)–(1.5) are supplemented with the following initial and boundary conditions:

$$(u, \mathbf{w}, \mathbf{b}, \theta_x)|_{\partial\Omega} = 0, \quad (1.6)$$

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$$(\rho, u, \mathbf{w}, \mathbf{b}, \theta)|_{t=0} = (\rho_0(x), u_0(x), \mathbf{w}_0(x), \mathbf{b}_0(x), \theta_0(x)), \quad (1.7)$$

where  $\partial\Omega = \{0, 1\}$  denotes the boundary of the interval  $\Omega := (0, 1)$ . The conditions (1.6) mean that the boundary is non-slip and thermally insulated.

There have been a lot of studies on MHD by physicists and mathematicians due to its physical importance, complexity, rich phenomena, and mathematical challenges. Below we mention some mathematical results on the compressible MHD equations, the interested readers can refer [3, 23, 30, 33, 37] for complete discussions on physical aspects. We begin with the one-dimensional case. The existence and uniqueness of local smooth solutions were proved firstly in [42], while the existence of global smooth solutions with small smooth initial data was shown in [25]. The exponential stability of small smooth solutions was obtained in [35, 38]. In [13, 40], Hoff and Tsyganov obtained the global existence and uniqueness of weak solutions with small initial energy. Under the technical condition that  $\kappa(\rho, \theta)$ , depending the temperature  $\theta$  only, i.e.,  $\kappa(\rho, \theta) \equiv \kappa(\theta)$ , satisfies

$$C^{-1}(1 + \theta^q) \leq \kappa(\theta) \leq C(1 + \theta^q), \quad (1.8)$$

for some  $q \geq 2$ , Chen and Wang [4] proved the existence, uniqueness, and Lipschitz continuous dependence of global strong solutions to the system (1.1)–(1.5) with large initial data satisfying

$$0 < \inf_{x \in \Omega} \rho_0(x) \leq \rho_0(x) \leq \sup_{x \in \Omega} \rho_0(x) < \infty; \quad \rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0 \in H^1(\Omega), \quad \inf_{x \in \Omega} \theta_0 > 0.$$

The similar results are obtained in [5, 43] for real gas cases. Recently, Fan, Jiang and Nakamura [8] obtained the global weak solutions to the problem (1.1)–(1.5) when the initial data satisfying the condition (1.8) with some  $q \geq 1$  and

$$\rho_0^{-1}, \rho_0 \in L^\infty(\Omega); \quad \rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0 \in L^2(\Omega); \quad \theta_0 \in L^1(\Omega), \quad \inf_{x \in \Omega} \theta_0 > 0.$$

Later they [9] obtained the existence, the uniqueness and the Lipschitz continuous dependence on the initial data of global weak solutions to the problem (1.1)–(1.7) when the initial data lie in the Lebesgue spaces.

For the multi-dimensional compressible MHD equations, there are also many mathematical results. As mentioned before, Vol’pert and Hudjaev [42] first obtained the local smooth solutions to the compressible MHD equations. Li, Su and Wang [32] obtained the existence and uniqueness of local in time strong solution with large initial data when the initial density has an positive lower bound. Fan and Yu [11] obtained the strong solution to the compressible MHD equations with vacuum. Kawashima [24] obtained the smooth solutions for two-dimensional compressible MHD equations when the initial data is a small perturbation of given constant state. Umeda, Kawashima and Shizuta [41] obtained the decay of solutions to the linearized MHD equations. Li and Yu [34] obtained the optimal decay rate of small smooth solutions. In [14, 15], Hu and Wang obtained the global existence of weak solutions to the isentropic compressible MHD equations and variational solutions to the full compressible MHD equations, see also [7, 10, 47] for related results. Suen and Hoff [39] obtained the global low-energy weak solutions of the isentropic compressible MHD equations. Xu and Zhang [46] obtained a blow-up criterion to the isentropic compressible MHD equations. We mention that the low Mach limit to the compressible MHD equations is an very important topic, and the interested reader can refer [16, 19–22, 28, 29, 31, 36] and the references cited therein.

It should be point out that although there are many progress on compressible MHD equations it is still an open question to obtain the global strong or smooth solutions to the full compressible MHD equations with large initial data and possible vacuum even in the one dimensional case, see [15].

In the present paper we study the global existence and uniqueness of large strong solutions to the planner compressible magnetohydrodynamic equations (1.1)-(1.5) with large initial data and vacuum. We focus on the perfect gas case:

$$P(\rho, \theta) := R\rho\theta, \quad e := C_V\theta,$$

where  $R > 0$  is the gas constant and  $C_V > 0$  is the heat capacity of the gas at constant volume. We will consider the case that the coefficients  $\lambda$ ,  $\mu$ , and  $\nu$  are positive constants and the heat conductivity coefficient depending the temperature  $\theta$  only, i.e.,  $\kappa(\rho, \theta) \equiv \kappa(\theta)$ . Since the positive physical constants  $\lambda, \mu, \nu, R$ , and  $C_V$  do not create essential mathematical difficulties in our analysis, we normalize them to be one for notational simplicity.

The main result in this paper reads as follows.

**Theorem 1.1.** *Let  $\kappa \in C^2[0, \infty)$  satisfy the condition (1.8) for some  $q > 0$ . Suppose that the initial data  $(\rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)$  satisfy  $\rho_0 \geq 0$ ,  $\theta_0 \geq 0$ ,  $\rho_0 \in H^2(\Omega)$ ,  $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ ,  $\mathbf{w}_0, \mathbf{b}_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ ,  $\theta_0 \in H^2(\Omega)$ ,  $(\theta_0)_x|_{\partial\Omega} = 0$  and the following compatible conditions*

$$\begin{cases} (u_0)_{xx} - (\rho_0\theta_0 + \frac{1}{2}|\mathbf{b}_0|^2)_x = \sqrt{\rho_0} g_1, \\ (\mathbf{w}_0)_{xx} - (\mathbf{b}_0)_x = \sqrt{\rho_0} \mathbf{g}_2, \\ (\kappa(\theta_0)(\theta_0)_x)_x + [(u_0)_x]^2 + |(\mathbf{w}_0)_x|^2 + |(\mathbf{b}_0)_x|^2 = \sqrt{\rho_0} g_3, \end{cases} \quad (1.9)$$

for some  $g_1, \mathbf{g}_2, g_3 \in L^2(\Omega)$ . Then for any  $T > 0$  there exists a unique global solution  $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$  to the problem (1.1)-(1.7) such that

$$\begin{aligned} \rho &\in L^\infty([0, T]; H^2(\Omega)), \quad \rho_t \in L^\infty([0, T]; H^1(\Omega)), \\ (u, \mathbf{w}, \mathbf{b}, \theta) &\in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \\ (\sqrt{\rho}u_t, \sqrt{\rho}\mathbf{w}_t, \mathbf{b}_t, \sqrt{\rho}\theta_t) &\in L^\infty(0, T; L^2(\Omega)), \\ (u_t, \mathbf{w}_t, \mathbf{b}_t, \theta_t) &\in L^2(0, T; H^1(\Omega)). \end{aligned}$$

There are two ingredients in our result comparing with the previous results on one-dimensional MHD equations mentioned above. First, in our result the initial density may contains the vacuum provided that it satisfies the compatible conditions (1.9). Next, a relaxed condition on the heat conductivity coefficient is permitted. In fact, in (1.8), we only need  $q > 0$  while in [4, 5, 43] required  $q \geq 2$  and in [8] with  $q \geq 1$ .

**Remark 1.1.** *It is possible to extend our results to the one-dimensional compressible MHD system with more general state of equations:*

$$P = \rho^2 \frac{\partial e}{\partial \rho} + \theta \frac{\partial P}{\partial \theta}$$

with some additional assumptions.

**Remark 1.2.** *When there is no vacuum initially, we can improve the results in [8] to the case  $q > 0$  by applying the arguments developed here.*

We remark that when taking  $\mathbf{w} = \mathbf{b} = 0$  the system (1.1)–(1.7) reduces to the well-known one-dimensional full Navier-Stokes equations and there are a lot of studies on this system. In the case of that the initial density is bounded away from zero, Kazhikhov and Shelukhin [27] first obtained the global smooth solutions for large initial data three decades ago, see also [1, 26, 48, 49] for different extensions. Recently, Huang and Li [18] obtain the global smooth solutions to the full Navier-Stokes system with possible vacuum and large oscillations provided that the total initial energy is sufficient small. Wen and Zhu obtained the global smooth solutions to the one-dimensional full Navier-Stokes system [44] and symmetric higher dimensional full Navier-Stokes system [45] with large initial data. For the variational or weak solutions to the full Navier-Stokes system, see [2, 12].

We give a few words on the strategy of the proof. Since the initial data may contains the vacuum we first construct the regularized initial density  $\rho_{0\delta}(x) = \rho_0 + \delta$  for any  $\delta > 0$ . Next, for each fixed  $\delta$ , we can obtain the local and uniqueness existence of strong solutions. Third, we establish sufficient *a priori* estimates uniformly with  $\delta$ . Combining the local existence result, and the uniformly *a priori* estimates, we obtain the desired global existence result by taking the limiting as  $\delta \rightarrow 0^+$  and applying standard continuity argument. We remark that the key point in the whole proof is to obtain uniformly *a priori* estimates where some ideas developed in [44, 45] are adapted. Comparing with [44], the main additional difficulties are due to the presence of the magnetic field and its interaction with the hydrodynamic motion of the flow of large oscillation. We shall deal with the terms involving the magnetic field very carefully, see especially Lemmas 2.5–2.7 below.

Before leaving this introduction we recall the following auxiliary inequalities.

**Lemma 1.2** ([12, 44]). *Let  $\Omega = (0, 1)$  be an interval in  $\mathbb{R}^1$ .*

(i). *Assume that  $\rho$  is a non-negative function satisfying*

$$0 < M \leq \int_{\Omega} \rho dx \leq K,$$

*for two constants  $M$  and  $K$ . Then, for any  $v \in H^1(\Omega)$ , it holds*

$$\|v\|_{L^\infty(\Omega)} \leq \frac{K}{M} \|v_x\|_{L^2(\Omega)} + \frac{1}{M} \left| \int_{\Omega} \rho v dx \right|,$$

(ii) *Assume further that  $v$  satisfies*

$$\|\rho v\|_{L^1(\Omega)} \leq \bar{C}.$$

*Then for any  $r > 0$ , there exists a positive constant  $C = C(M, K, r, \bar{C})$  such that*

$$\|v^r\|_{L^\infty(\Omega)} \leq C \|(v^r)_x\|_{L^2(\Omega)} + C.$$

## 2. PROOF OF THEOREM 1.1

In this section we will prove Theorem 1.1 by consider the initial density  $\rho_{0\delta} = \rho_0 + \delta$ , as mentioned before, to get a sequence of approximate solutions to (1.1)–(1.7), then taking  $\delta \rightarrow 0^+$  after making some *a priori* estimates uniformly for  $\delta$ . Since the proof of the local existence and uniqueness of strong solutions to the approximate problem is now standard [6, 11], thus we only need to establish the uniform estimates.

Below we still use  $(\rho, u, \mathbf{w}, \mathbf{b}, \theta)$  to denote the smooth solutions of approximate problem to (1.1)–(1.7). We shall denote  $Q_T := \Omega \times [0, T]$  with  $T > 0$  and omit the spatial domain  $\Omega$  in integrals for convenience. We use  $C$  to denote the constants which are independent of  $\delta$  and may change from line to line.

To begin with the proof, we notice that the total mass and energy in the system (1.1)–(1.5) are conserved. In fact, by rewriting (1.1)–(1.5) one has

$$\mathcal{E}_t + \left[ u \left( \mathcal{E} + P + \frac{1}{2} |\mathbf{b}|^2 \right) - \mathbf{w} \cdot \mathbf{b} \right]_x = (uu_x + \mathbf{w} \cdot \mathbf{w}_x + \mathbf{b} \cdot \mathbf{b}_x + \kappa(\theta)\theta_x)_x, \quad (2.1)$$

$$(\rho\mathcal{S})_t + (\rho u\mathcal{S})_x - \left( \frac{\kappa(\theta)}{\theta} \theta_x \right)_x = \frac{u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2}{\theta} + \frac{\kappa(\theta)\theta_x^2}{\theta^2}, \quad (2.2)$$

where  $\mathcal{E}$  and  $\mathcal{S}$  are the total energy and the entropy, respectively,

$$\mathcal{E} := \rho \left( \theta + \frac{1}{2} (u^2 + |\mathbf{w}|^2) \right) + \frac{1}{2} |\mathbf{b}|^2, \quad \mathcal{S} := \ln \theta - \ln \rho.$$

Integrating (1.1), (2.1) and (2.2) over  $Q_T$ , we have

**Lemma 2.1.**

$$\begin{aligned} \int \rho(x, t) dx &= \int \rho_0(x) dx, & \int \mathcal{E}(x, t) dx &= \int \mathcal{E}(x, t=0) dx, \\ \int (\rho \ln \rho + \rho |\ln \theta|)(x, t) dx &+ \iint_{Q_T} \left( \frac{u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2}{\theta} + \frac{\kappa(\theta)\theta_x^2}{\theta^2} \right) dx dt \leq C. \end{aligned}$$

**Lemma 2.2.**

$$0 \leq \rho(x, t) \leq C, \quad (x, t) \in \overline{Q_T}. \quad (2.3)$$

*Proof.* We need only to estimate the upper bound. Here we borrow the proof from [8] for reader's convenience. From (1.1) and (1.2), we have

$$(\rho u)_t = \tilde{P}_x, \quad \tilde{P}_x := u_x - \rho u^2 - P - \frac{1}{2} |\mathbf{b}|^2.$$

Denote

$$\phi := \int_0^t \tilde{P}(x, \tau) d\tau + \int_0^x \rho(\xi) u_0(\xi) d\xi,$$

we have

$$\phi_x = \rho u, \quad \phi_t = \tilde{P}, \quad \phi_x|_{\partial\Omega} = 0, \quad \phi|_{t=0} = \int_0^x \rho_0(\xi) u_0(\xi) d\xi. \quad (2.4)$$

By virtue of Lemma 2.1 and Cauchy inequality, it holds

$$\|\phi_x\|_{L^\infty(0, T; L^1(\Omega))} \leq C, \quad \left| \int \phi dx \right| \leq C.$$

Hence

$$\|\phi\|_{L^\infty(0, T; L^\infty(\Omega))} \leq C. \quad (2.5)$$

Now, denoting  $F := e^\phi$  and using (2.4), we have after a straightforward calculation that

$$D_t(\rho F) := \partial_t(\rho F) + u \partial_x(\rho F) = - \left( p + \frac{1}{2} |\mathbf{b}|^2 \right) \rho F \leq 0,$$

which together with (2.5) implies (2.3) immediately.  $\square$

**Lemma 2.3.** *Let  $0 < \alpha < \min\{1, q\}$  be any given constant, then it holds*

$$\iint_{Q_T} \left( \frac{u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2}{\theta^\alpha} + \frac{(1 + \theta^q)\theta_x^2}{\theta^{1+\alpha}} \right) \leq C, \quad (2.6)$$

$$\int_0^T \|\theta\|_{L^\infty}^{q-\alpha+1} \leq C. \quad (2.7)$$

*Proof.* The proof is similar to that given in [45] for symmetric Navier-Stokes equations, we present it here for completeness. Multiplying (1.5) by  $\theta^{-\alpha}$  and integrating the result over  $Q_T$ , we have

$$\begin{aligned} \iint_{Q_T} \frac{u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2}{\theta^\alpha} dxdt + \alpha \iint_{Q_T} \frac{\kappa(\theta)\theta_x^2}{\theta^{1+\alpha}} dxdt \\ = \iint_{Q_T} \{(\rho\theta)_t + (\rho u\theta)_x + Pu_x\} \theta^{-\alpha} dxdt. \end{aligned} \quad (2.8)$$

Using (1.1) and Lemma 2.1, the first two terms on the right-hand side of (2.8) can be bounded as

$$\begin{aligned} \iint_{Q_T} \{(\rho\theta)_t + (\rho u\theta)_x\} \theta^{-\alpha} dxdt &= \int_\Omega \rho \int_0^\theta \frac{1}{\xi^\alpha} d\xi dxdt - \int_\Omega \rho_0 \int_0^{\theta_0} \frac{1}{\xi^\alpha} d\xi dx \\ &= \int_\Omega \frac{1}{1-\alpha} \rho \theta^{1-\alpha} dxdt - \int_\Omega \frac{1}{1-\alpha} \rho_0 \theta_0^{1-\alpha} dx \\ &\leq C + C \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt. \end{aligned} \quad (2.9)$$

By Cauchy inequality, Lemmas 2.1 and 2.2, we have

$$\begin{aligned} \iint_{Q_T} Pu_x \theta^{-\alpha} dxdt &\leq \frac{1}{2} \iint_{Q_T} \frac{u_x^2}{\theta^\alpha} dxdt + C \iint_{Q_T} \rho^2 \theta^{2-\alpha} dxdt \\ &\leq \frac{1}{2} \iint_{Q_T} \frac{u_x^2}{\theta^\alpha} dxdt + C \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt. \end{aligned} \quad (2.10)$$

Noticing  $0 < \alpha < \min\{1, q\}$ , the Hölder inequality and Lemma 1.2 imply that

$$\begin{aligned} C \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt &\leq C + \int_0^T \|\theta^{-\alpha} \theta_x\|_{L^2} dt \\ &\leq C + C \int_0^T \left( \int \frac{\theta^2 \theta^{1-\alpha}}{\theta^{1+\alpha}} \right)^{1/2} dt \\ &\leq C + \frac{\alpha}{2} \iint_{Q_T} \frac{\kappa(\theta)\theta_x^2}{\theta^{1+\alpha}} dxdt \end{aligned} \quad (2.11)$$

for  $q \geq 1 - \alpha$ , while

$$\begin{aligned} C \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt &\leq C + \int_0^T \|\theta^{-\alpha} \theta_x\|_{L^2} dt \\ &\leq C + C \int_0^T \left( \int \frac{\theta^q \theta_x^2}{\theta^{1+\alpha}} \theta^{1-\alpha-q} \right)^{1/2} dt \\ &\leq C + \frac{\alpha}{2} \iint_{Q_T} \frac{\kappa(\theta)\theta_x^2}{\theta^{1+\alpha}} dxdt + \tilde{C} \int_0^T \|\theta\|_{L^\infty}^{1-\alpha-q} dt \end{aligned}$$

$$\leq C + \frac{\alpha}{2} \iint_{Q_T} \frac{\kappa(\theta)\theta_x^2}{\theta^{1+\alpha}} dx dt + \frac{C}{2} \int_0^T \|\theta\|_{L^\infty}^{1-\alpha} dt \quad (2.12)$$

for  $0 < q < 1 - \alpha$ .

Putting (2.8)-(2.12) into (2.7) implies (2.6). By Lemma 1.2 and (2.7), we can easily get (2.7).  $\square$

**Lemma 2.4.**

$$\iint_{Q_T} (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) dx dt \leq C. \quad (2.13)$$

*Proof.* Multiplying (1.2) by  $u$ , using (1.1), and integrating the result over  $\Omega$ , we see that

$$\frac{1}{2} \frac{d}{dt} \int \rho u^2 dx + \int u_x^2 dx = \int P u_x dx - \int u \mathbf{b} \cdot \mathbf{b}_x dx. \quad (2.14)$$

Similar, multiplying (1.3) by  $\mathbf{w}$ , using (1.1), and integrating the result over  $\Omega$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \int \rho |\mathbf{w}|^2 dx + \int |\mathbf{w}_x|^2 dx = \int \mathbf{b}_x \cdot \mathbf{w} dx = - \int \mathbf{b} \cdot \mathbf{w}_x dx. \quad (2.15)$$

Multiplying (1.4) by  $\mathbf{b}$  and then integrating them over  $\Omega$ , we infer that

$$\frac{1}{2} \frac{d}{dt} \int |\mathbf{b}|^2 dx + \int |\mathbf{b}_x|^2 dx = \int \mathbf{b} \mathbf{w}_x dx + \int u \mathbf{b} \mathbf{b}_x dx. \quad (2.16)$$

Summing up (2.14), (2.15) and (2.16), using (2.3) and (2.7), we get

$$\frac{1}{2} \frac{d}{dt} \int (\rho u^2 + \rho |\mathbf{w}|^2 + |\mathbf{b}|^2) dx + \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) dx = \int P u_x dx. \quad (2.17)$$

By Young inequality, Lemmas 2.1 and 2.2, the estimate (2.7), and the condition that  $0 < \alpha < \min\{1, q\}$ , we have

$$\begin{aligned} \iint_{Q_T} P u_x dx dt &\leq C \iint_{Q_T} \rho^2 \theta^2 dx dt + \frac{1}{2} \int_0^T \|u_x\|_{L^2}^2 dt \\ &\leq C \int_0^T \|\theta\|_{L^\infty} dt + \frac{1}{2} \int_0^T \|u_x\|_{L^2}^2 dt \\ &\leq C \int_0^T \|\theta\|_{L^\infty}^{q-\alpha+1} dt + \frac{1}{2} \int_0^T \|u_x\|_{L^2}^2 dt \\ &\leq C + \frac{1}{2} \int_0^T \|u_x\|_{L^2}^2 dt. \end{aligned}$$

Integrating (2.17) over  $[0, T]$  and applying the above inequality give (2.13).  $\square$

**Lemma 2.5.**

$$\begin{aligned} &\int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2 + P^2 + \rho \theta^{q+2}) dx \\ &+ \iint_{Q_T} (\rho u_t^2 + \rho |\mathbf{w}_t|^2 + |\mathbf{b}_t|^2 + |\mathbf{b}_{xx}|^2 + \kappa^2(\theta) \theta_x^2) dx dt \leq C. \end{aligned} \quad (2.18)$$

*Proof.* Multiplying (1.2) by  $u_t$ , and then integrating them over  $\Omega$ , we infer that

$$\frac{1}{2} \frac{d}{dt} \int u_x^2 dx + \int \rho u_t^2 dx + \int \rho u u_x u_t dx$$

$$\begin{aligned}
&= - \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right)_x u_t dx \\
&= \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right) u_{xt} dx \\
&= \frac{d}{dt} \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right) u_x dx - \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right)_t u_x dx \\
&= \frac{d}{dt} \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right) u_x dx - \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right)_t \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right) dx \\
&\quad - \frac{1}{2} \frac{d}{dt} \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right)^2 dx. \tag{2.19}
\end{aligned}$$

First, by Cauchy inequality and Poincaré inequality, it is easy to find that

$$\begin{aligned}
\left| \int \rho u u_x u_t dx \right| &\leq \frac{1}{16} \int \rho u_t^2 dx + C \int \rho u^2 u_x^2 dx \\
&\leq \frac{1}{16} \int \rho u_t^2 dx + C \|\rho\|_{L^\infty} \|u\|_{L^\infty}^2 \int u_x^2 dx \\
&\leq C + \frac{1}{16} \int \rho u_t^2 dx + C \left( \int u_x^2 dx \right)^2. \tag{2.20}
\end{aligned}$$

From (1.4) and (1.5), we have

$$\left( \frac{1}{2} |\mathbf{b}|^2 \right)_t + \mathbf{b} \cdot (u \mathbf{b} - \mathbf{w})_x = (\mathbf{b} \cdot \mathbf{b}_x)_x - |\mathbf{b}_x|^2, \tag{2.21}$$

$$P_t + (uP - \kappa(\theta)\theta_x)_x = u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2 - Pu_x. \tag{2.22}$$

Using (2.21), (2.22), (1.2), (2.3), and Lemma 2.2, we obtain

$$\begin{aligned}
&- \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right)_t \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right) dx \\
&= - \int [u_x^2 + |\mathbf{w}_x|^2 - Pu_x - \mathbf{b} \cdot (u \mathbf{b} - \mathbf{w})_x] \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right) dx \\
&\quad + \int (\kappa(\theta)\theta_x - uP + \mathbf{b} \cdot \mathbf{b}_x) \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right)_x dx \\
&= - \int [u_x^2 + |\mathbf{w}_x|^2 - Pu_x - \mathbf{b} \cdot (u \mathbf{b} - \mathbf{w})_x] \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right) dx \\
&\quad + \int (\kappa(\theta)\theta_x - uP + \mathbf{b} \cdot \mathbf{b}_x) (\rho u_t + \rho u u_x) dx \\
&\leq \int |u_x^2 + |\mathbf{w}_x|^2 - Pu_x - \mathbf{b} \cdot (u \mathbf{b} - \mathbf{w})_x| dx \left\| u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right\|_{L^\infty} \\
&\quad + \|\kappa(\theta)\theta_x - uP + \mathbf{b} \cdot \mathbf{b}_x\|_{L^2} \left( \|\sqrt{\rho} u_t\|_{L^2} \|\rho\|_{L^1}^{1/2} + \|\rho\|_{L^\infty} \|u\|_{L^\infty} \|u_x\|_{L^2} \right) \\
&\leq C \left( \left\| u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right\|_{L^1} + \left\| \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right)_x \right\|_{L^1} \right) \\
&\quad \times \int (u_x^2 + |\mathbf{w}_x|^2 + P^2 + |\mathbf{b}|^2 + |\mathbf{b}|^4 + u^2 + |\mathbf{b}_x|^2) dx \\
&\quad + C \|\kappa(\theta)\theta_x - uP + \mathbf{b} \cdot \mathbf{b}_x\|_{L^2} (\|\sqrt{\rho} u_t\|_{L^2} + C \|u_x\|_{L^2}^2)
\end{aligned}$$



$$\begin{aligned}
&\leq C \int (u_x^2 + |\mathbf{w}_x|^2 + P^2 + |\mathbf{b}_x|^2) dx \left\{ 1 + \|u_x\|_{L^2} + \|\rho u_t + \rho u u_x\|_{L^1} \right\} \\
&\quad + C \|\kappa(\theta)\theta_x - uP + \mathbf{b} \cdot \mathbf{b}_x\|_{L^2} (\|\sqrt{\rho}u_t\|_{L^2} + C\|u_x\|_{L^2}^2) \\
&\leq C \left\{ 1 + \|u_x\|_{L^2}^4 + \|\mathbf{w}_x\|_{L^2}^4 + \|\mathbf{b}_x\|_{L^2}^4 + \|\kappa(\theta)\theta_x\|_{L^2}^2 \right\} + \frac{1}{16} \|\sqrt{\rho}u_t\|_{L^2}^2. \quad (2.23)
\end{aligned}$$

Multiplying (1.3) by  $\mathbf{w}_t$ , using (2.3), and integrating the result over  $\Omega$ , we derive

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int |\mathbf{w}_x|^2 dx + \int \rho |\mathbf{w}_t|^2 dx \\
&= \int \mathbf{b}_x \cdot \mathbf{w}_t dx - \int \rho u \mathbf{w}_x \cdot \mathbf{w}_t dx \\
&= - \int \mathbf{b} \cdot \mathbf{w}_{xt} dx - \int \rho u \mathbf{w}_x \cdot \mathbf{w}_t dx \\
&= - \frac{d}{dt} \int \mathbf{b} \cdot \mathbf{w}_x dx + \int \mathbf{b}_t \cdot \mathbf{w}_x dx - \int \rho u \mathbf{w}_x \cdot \mathbf{w}_t dx \\
&\leq - \frac{d}{dt} \int \mathbf{b} \cdot \mathbf{w}_x dx + \frac{1}{16} \int |\mathbf{b}_t|^2 dx + C \int |\mathbf{w}_x|^2 dx \\
&\quad + \frac{1}{16} \int \rho |\mathbf{w}_t|^2 dx + C \|u_x\|_{L^2}^4 + C \|\mathbf{w}_x\|_{L^2}^4. \quad (2.24)
\end{aligned}$$

Multiplying (1.4) by  $\mathbf{b}_t - \mathbf{b}_{xx}$ , integrating the result over  $\Omega$ , and using Cauchy inequality, we have

$$\begin{aligned}
&\frac{d}{dt} \int |\mathbf{b}_x|^2 dx + \int (|\mathbf{b}_t|^2 + |\mathbf{b}_{xx}|^2) dx \\
&= \int (\mathbf{w} - u\mathbf{b})_x \cdot (\mathbf{b}_t - \mathbf{b}_{xx}) dx \\
&\leq C \int |\mathbf{w}_x|^2 dx + C \|u_x\|_{L^2}^4 + C \|\mathbf{b}_x\|_{L^2}^4 + \frac{1}{16} \int (|\mathbf{b}_t|^2 + |\mathbf{b}_{xx}|^2) dx. \quad (2.25)
\end{aligned}$$

Multiplying (1.5) by  $\theta^{q+1}$ , using (1.8), and integrating the result over  $\Omega$ , we find that

$$\begin{aligned}
&\frac{d}{dt} \int \rho \theta^{q+2} dx + C \int \kappa^2 \theta_x^2 dx \\
&\leq C \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2 + P^2) \theta^{q+1} dx \\
&\leq C \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2 + P^2) dx \|\theta^{q+1}\|_{L^\infty} \\
&\leq C \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2 + P^2) dx (\|\kappa \theta_x\|_{L^2} + 1) \\
&\leq \frac{C}{16} \|\kappa \theta_x\|_{L^2}^2 + C \left( \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2 + P^2) dx \right)^2 + C, \quad (2.26)
\end{aligned}$$

where we used the estimate:

$$\|\theta\|_{L^\infty}^{q+1} \leq C(1 + \|\kappa(\theta)\theta_x\|_{L^2}) \quad (2.27)$$

which can be derived from Lemma 1.2 and (1.8).

Combining (2.19), (2.20), (2.23), (2.24), and (2.25) with (2.26), we arrive at (2.18).  $\square$

**Lemma 2.6.**

$$\int \rho_x^2 + \rho_t^2 dx + \iint_{Q_T} u_{xx}^2 + |\mathbf{w}_{xx}|^2 dx dt \leq C. \quad (2.28)$$

*Proof.* Applying the operator  $\partial_x$  to (1.1) gives

$$\rho_{xt} + \rho_{xx}u + 2\rho_x u_x + \rho u_{xx} = 0.$$

Multiplying the above equation by  $2\rho_x$ , integrating the result over  $\Omega$ , and using (2.3) and (2.18), we find that

$$\begin{aligned} \frac{d}{dt} \int \rho_x^2 dx &= -3 \int \rho_x^2 u_x dx - 2 \int \rho \rho_x u_{xx} dx \\ &= -3 \int \rho_x^2 \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right) dx - 3 \int \rho_x^2 \left( p + \frac{1}{2} |\mathbf{b}|^2 \right) dx \\ &\quad - 2 \int \rho \rho_x u_{xx} dx \\ &\leq -3 \int \rho_x^2 \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right) dx \\ &\quad - 2 \int \rho \rho_x u_{xx} dx \\ &\leq 3 \left\| u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right\|_{L^\infty} \int \rho_x^2 dx + C \|\rho\|_{L^\infty} \|\rho_x\|_{L^2} \|u_{xx}\|_{L^2} \\ &\leq C \left( \left\| u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right\|_{L^2} + \left\| \left( u_x - P - \frac{1}{2} |\mathbf{b}|^2 \right)_x \right\|_{L^2} \right) \int \rho_x^2 dx \\ &\quad + C \|\rho_x\|_{L^2} \|u_{xx}\|_{L^2} \\ &\leq C(1 + \|\sqrt{\rho} u_t\|_{L^2}) \|\rho_x\|_{L^2}^2 + C \|u_{xx}\|_{L^2}^2. \end{aligned} \quad (2.29)$$

On the other hand, it follows from (1.2) that

$$\begin{aligned} \|u_{xx}\|_{L^2} &\leq C(\|\sqrt{\rho} u_t\|_{L^2} + \|\rho u u_x\|_{L^2} + \|P_x\|_{L^2} + \|\mathbf{b} \cdot \mathbf{b}_x\|_{L^2}) \\ &\leq C(1 + \|\sqrt{\rho} u_t\|_{L^2} + \|u_x\|_{L^2}^2 + \|\theta_x\|_{L^2} + \|\rho_x\|_{L^2} \|\theta\|_{L^\infty}) \\ &\leq C(1 + \|\sqrt{\rho} u_t\|_{L^2} + \|u_x\|_{L^2}^2 + \|\kappa(\theta) \theta_x\|_{L^2} + \|\rho_x\|_{L^2} \|\theta\|_{L^\infty}). \end{aligned} \quad (2.30)$$

Inserting (2.30) into (2.29), using Lemmas 2.3 and 2.5, (2.27), and the Gronwall inequality, we have

$$\|\rho_x\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \quad (2.31)$$

It follows from (1.1), (2.3), (2.18) and (2.31) that

$$\|\rho_t\|_{L^\infty(0,T;L^2(\Omega))} \leq C.$$

Thus (2.30) yields

$$\|u_{xx}\|_{L^2(0,T;L^2(\Omega))} \leq C.$$

It follows from (1.3), (2.3), and (2.18) that

$$\|\mathbf{w}_{xx}\|_{L^2(0,T;L^2(\Omega))} \leq C.$$

□

**Lemma 2.7.**

$$\int (\rho u_t^2 + \rho |\mathbf{w}_t|^2 + |\mathbf{b}_t|^2 + u_{xx}^2 + |\mathbf{w}_{xx}|^2 + |\mathbf{b}_{xx}|^2 + \kappa^2(\theta) \theta_x^2) dx$$

$$+ \iint_{Q_T} (u_{xt}^2 + |\mathbf{w}_{xt}|^2 + |\mathbf{b}_{xt}|^2 + \rho\theta_t^2 + \theta_{xx}^2) dx dt \leq C. \quad (2.32)$$

*Proof.* Applying  $\partial_t$  to (1.2), we see that

$$\rho u_{tt} + \rho u u_{xt} - u_{xxt} = - \left( P + \frac{1}{2} |\mathbf{b}|^2 \right)_{xt} - \rho_t u_t - \rho_t u u_x - \rho u_t u_x.$$

Multiplying the above equation by  $u_t$ , integrating them over  $\Omega$ , and using (1.1), Lemmas 2.5 and 2.6, and Cauchy inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho u_t^2 dx + \int u_{xt}^2 dx \\ &= \int \left( P + \frac{1}{2} |\mathbf{b}|^2 \right)_t u_{xt} dx - 2 \int \rho u u_t u_{xt} dx - \int \rho_t u u_x u_t dx - \int \rho u_t^2 u_x dx \\ &\leq \int (\rho \theta_t + \theta \rho_t + \mathbf{b} \cdot \mathbf{b}_t) u_{xt} dx + 2 \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho}\|_{L^\infty} \|u\|_{L^\infty} \|u_{xt}\|_{L^2} \\ &\quad + \|\rho_t\|_{L^2} \|u\|_{L^\infty} \|u_x\|_{L^2} \|u_t\|_{L^\infty} + \|u_x\|_{L^\infty} \int \rho u_t^2 dx \\ &\leq \epsilon_1 \int u_{xt}^2 dx + C \int \rho \theta_t^2 dx + \|\theta\|_{L^\infty}^2 + \|\mathbf{b}_t\|_{L^2}^2 \\ &\quad + C \int \rho u_t^2 dx + \|u_{xx}\|_{L^2} \int \rho u_t^2 dx + C \end{aligned} \quad (2.33)$$

for any  $0 < \epsilon_1 < 1$ .

Testing (1.5) by  $\kappa(\theta)\theta_t = \left( \int_0^\theta k(\xi) d\xi \right)_t$ , using (1.1), Lemma 2.5, Lemma 2.6, we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \kappa^2(\theta) \theta_x^2 dx + \int \rho \kappa(\theta) \theta_t^2 dx \\ &= - \int \rho u \theta_x \kappa(\theta) \theta_t dx - \int \rho \theta u_x \kappa(\theta) \theta_t dx \\ &\quad + \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) \left( \int_0^\theta \kappa(\xi) d\xi \right)_t dx \\ &\leq \epsilon_2 \int \rho \kappa(\theta) \theta_t^2 dx + C \int \rho u^2 \kappa(\theta) \theta_x^2 dx + C \int \rho \theta^2 \kappa(\theta) u_x^2 dx \\ &\quad + \frac{d}{dt} \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) \int_0^\theta \kappa(\xi) d\xi dx \\ &\quad - 2 \int (u_x u_{xt} + \mathbf{w}_x \cdot \mathbf{w}_{xt} + \mathbf{b}_x \cdot \mathbf{b}_{xt}) \int_0^\theta \kappa(\xi) d\xi dx \\ &\leq \epsilon_2 \int \rho \kappa(\theta) \theta_t^2 dx + C \int \kappa \theta_x^2 dx + C \|u_x\|_{L^\infty}^2 \int \rho \theta^2 \kappa dx \\ &\quad + \frac{d}{dt} \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) \int_0^\theta \kappa(\xi) d\xi dx \\ &\quad + C (\|u_x\|_{L^2} \|u_{xt}\|_{L^2} + \|\mathbf{w}_x\|_{L^2} \|\mathbf{w}_{xt}\|_{L^2} + \|\mathbf{b}_x\|_{L^2} \|\mathbf{b}_{xt}\|_{L^2}) \left\| \int_0^\theta \kappa(\xi) d\xi \right\|_{L^\infty} \\ &\leq \epsilon_2 \int \rho \kappa(\theta) \theta_t^2 dx + C \int \kappa(\theta) \theta_x^2 dx + C (1 + \|u_{xx}\|_{L^2}^2) \int \rho \theta^2 (1 + \theta^q) dx \end{aligned}$$

$$\begin{aligned}
& + \frac{d}{dt} \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) \int_0^\theta \kappa(\xi) d\xi dx \\
& + C(\|u_{xt}\|_{L^2} + \|\mathbf{w}_{xt}\|_{L^2} + \|\mathbf{b}_{xt}\|_{L^2}) \|\theta(1 + \theta^q)\|_{L^\infty} \\
& \leq \epsilon_2 \int \rho \kappa(\theta) \theta_t^2 dx + \epsilon_3 (\|u_{xt}\|_{L^2}^2 + \|\mathbf{w}_{xt}\|_{L^2}^2 + \|\mathbf{b}_{xt}\|_{L^2}^2) \\
& + C \int \kappa(\theta) \theta_x^2 dx + C(1 + \|u_{xx}\|_{L^2}^2) \\
& + \frac{d}{dt} \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) \int_0^\theta \kappa(\xi) d\xi dx + C\|\theta(1 + \theta^q)\|_{L^\infty}^2, \tag{2.34}
\end{aligned}$$

for any  $0 < \epsilon_2, \epsilon_3 < 1$ .

Applying the operator  $\partial_t$  to (1.3) gives

$$\rho \mathbf{w}_{tt} + \rho u \mathbf{w}_{xt} - \mathbf{w}_{xxt} = -\rho_t \mathbf{w}_t - \rho_t u \mathbf{w}_x - \rho u_t \mathbf{w}_x + \mathbf{b}_{xt}. \tag{2.35}$$

Multiplying (2.35) by  $w_t$ , integrating the result over  $\Omega$ , and using (1.1), Lemmas 2.5 and 2.6, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \rho |\mathbf{w}_t|^2 dx + \int |\mathbf{w}_{xt}|^2 dx \\
& = -2 \int \rho u \mathbf{w}_t \cdot \mathbf{w}_{xt} dx - \int \rho_t u \mathbf{w}_x \cdot \mathbf{w}_t dx \\
& \quad - \int \rho u_t \mathbf{w}_x \cdot \mathbf{w}_t dx - \int \mathbf{b}_x \cdot \mathbf{w}_{xt} dx \\
& \leq 2 \|\sqrt{\rho} \mathbf{w}_t\|_{L^2} \|\sqrt{\rho} u\|_{L^\infty} \|\mathbf{w}_{xt}\|_{L^2} + \|\rho_t\|_{L^2} \|u\|_{L^\infty} \|\mathbf{w}_x\|_{L^2} \|\mathbf{w}_t\|_{L^\infty} \\
& \quad + \|\sqrt{\rho} \mathbf{w}_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\mathbf{w}_x\|_{L^\infty} + \|\mathbf{b}_x\|_{L^2} \|\mathbf{w}_{xt}\|_{L^2} \\
& \leq C \|\sqrt{\rho} \mathbf{w}_t\|_{L^2} \|\mathbf{w}_{xt}\|_{L^2} + C \|\mathbf{w}_t\|_{L^\infty} \\
& \quad + C \|\sqrt{\rho} \mathbf{w}_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\mathbf{w}_{xx}\|_{L^2} + C \|\mathbf{w}_{xt}\|_{L^2} \\
& \leq \epsilon_4 \|w_{xt}\|_{L^2}^2 + C + C \|\sqrt{\rho} \mathbf{w}_t\|_{L^2}^2 + C \|\mathbf{w}_{xx}\|_{L^2}^2 \|\sqrt{\rho} u_t\|_{L^2}^2, \tag{2.36}
\end{aligned}$$

for any  $0 < \epsilon_4 < 1$ .

Applying the operator  $\partial_t$  to (1.4) gives

$$\mathbf{b}_{tt} - \mathbf{b}_{xxt} = -(u \mathbf{b} - \mathbf{w})_{xt}.$$

Multiplying the above equation by  $\mathbf{b}_t$ , integrating the result over  $\Omega$ , and using Lemma 2.5, we find that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \mathbf{b}_t^2 dx + \int \mathbf{b}_{xt}^2 dx \\
& = \int (u \mathbf{b} - \mathbf{w})_t \cdot \mathbf{b}_{xt} dx \\
& = \int (u_t \mathbf{b} + \mathbf{b}_t u - \mathbf{w}_t) \cdot \mathbf{b}_{xt} dx \\
& \leq (\|\mathbf{b}\|_{L^\infty} \|u_t\|_{L^2} + \|u\|_{L^\infty} \|\mathbf{b}_t\|_{L^2} + \|\mathbf{w}_t\|_{L^2}) \|\mathbf{b}_{xt}\|_{L^2} \\
& \leq C(\|u_t\|_{L^2} + \|\mathbf{b}_t\|_{L^2} + \|\mathbf{w}_t\|_{L^2}) \|\mathbf{b}_{xt}\|_{L^2} \\
& \leq C(\|u_{xt}\|_{L^2} + \|\mathbf{b}_t\|_{L^2} + \|\mathbf{w}_{xt}\|_{L^2}) \|\mathbf{b}_{xt}\|_{L^2} \\
& \leq \frac{1}{2} \|\mathbf{b}_{xt}\|_{L^2}^2 + C(\|u_{xt}\|_{L^2}^2 + \|\mathbf{w}_{xt}\|_{L^2}^2 + \|\mathbf{b}_t\|_{L^2}^2). \tag{2.37}
\end{aligned}$$

Combining (2.33), (2.34), (2.27), and (2.36) with (2.37), taking  $\epsilon_i$  ( $i = 1, \dots, 4$ ) small enough, integrating the resulting inequality over  $(0, t)$ , then we conclude that

$$\int \rho u_t^2 + \rho |\mathbf{w}_t|^2 + |\mathbf{b}_t|^2 + \kappa^2(\theta) \theta_x^2 dx + \iint_{Q_T} u_{xt}^2 + |\mathbf{w}_{xt}|^2 + |\mathbf{b}_{xt}|^2 + \rho \theta_t^2 dx dt \leq C. \quad (2.38)$$

where we have used the following estimate:

$$\begin{aligned} & \int_0^t \left( \frac{d}{dt} \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) \int_0^\theta \kappa(\xi) d\xi dx \right) d\tau \\ & \leq \int (u_x^2 + |\mathbf{w}_x|^2 + |\mathbf{b}_x|^2) dx \left\| \int_0^\theta \kappa(\xi) d\xi \right\|_{L^\infty} + C \\ & \leq C \left\| \int_0^\theta \kappa(\xi) d\xi \right\|_{L^\infty} + C \\ & \leq C \left\| \left( \int_0^\theta \kappa(\xi) d\xi \right)_x \right\|_{L^2} + C \\ & \leq C \|\kappa(\theta) \theta_x\|_{L^2} + C \leq \epsilon_5 \|\kappa(\theta) \theta_x\|_{L^2}^2 + C, \end{aligned}$$

for any  $0 < \epsilon_5 < 1$ .

It follows from (1.2), (1.3), (1.4), (2.38), and Lemmas 2.5 Lemma 2.6 that

$$\int u_{xx}^2 + |\mathbf{w}_{xx}|^2 + |\mathbf{b}_{xx}|^2 dx \leq C.$$

Noting the above estimate, (1.5), and (2.38), it follows that

$$\begin{aligned} \|\theta_{xx}\|_{L^2}^2 & \leq C \int (\theta_x^4 + u_x^4 + |\mathbf{w}_x|^4 + |\mathbf{b}_x|^4 + \rho \theta_t^2 + u^2 \theta_x^2 + \theta^2 u_x^2) dx \\ & \leq C + C \int \theta_x^4 dx + C \int \rho \theta_t^2 dx \\ & \leq C + C \|\theta_x^2\|_{L^\infty} \int \theta_x^2 dx + C \int \rho \theta_t^2 dx \\ & \leq C + C \|(\theta_x^2)_x\|_{L^1} + C \int \rho \theta_t^2 dx \\ & \leq C + C \|\theta_x \theta_{xx}\|_{L^1} + C \int \rho \theta_t^2 dx \\ & \leq C + C \|\theta_{xx}\|_{L^2} + C \int \rho \theta_t^2 dx, \end{aligned}$$

which yields

$$\|\theta_{xx}\|_{L^2}^2 \leq C + C \int \rho \theta_t^2 dx. \quad (2.39)$$

□

**Lemma 2.8.**

$$\int (\rho_{xx}^2 + \rho_{xt}^2) dx + \iint_{Q_T} (\rho_{tt}^2 + u_{xxx}^2) dx dt \leq C. \quad (2.40)$$

*Proof.* Applying the operator  $\partial_x^2$  to (1.1) gives

$$\rho_{xxt} = -\rho_{xxx}u - 3\rho_{xx}u_x - 3\rho_x u_{xx} - \rho u_{xxx}.$$

Multiplying the above equation by  $2\rho_{xx}$ , integrating them over  $\Omega$ , and using Lemmas 2.7 and 2.6, we find that

$$\begin{aligned} & \frac{d}{dt} \int \rho_{xx}^2 dx \\ &= -5 \int \rho_{xx}^2 u_x dx - 6 \int \rho_x \rho_{xx} u_{xx} dx - 2 \int \rho \rho_{xx} u_{xxx} dx \\ &\leq 5 \|u_x\|_{L^\infty} \int \rho_{xx}^2 dx + 6 \|\rho_x\|_{L^\infty} \|\rho_{xx}\|_{L^2} \|u_{xx}\|_{L^2} + 2 \|\rho\|_{L^\infty} \|\rho_{xx}\|_{L^2} \|u_{xxx}\|_{L^2} \\ &\leq C \int \rho_{xx}^2 dx + C \int u_{xxx}^2 dx + C. \end{aligned} \quad (2.41)$$

Applying  $\partial_x$  to (1.2), integrating them over  $\Omega$ , and using Lemmas 2.6 and 2.7, we infer that

$$\begin{aligned} & \|u_{xxx}\|_{L^2} \\ &\leq \left\| (\rho u)_{xt} + \left( \rho u^2 + P + \frac{1}{2} |\mathbf{b}|^2 \right)_{xx} \right\|_{L^2} \\ &\leq \left\| \rho_x u_t + \rho u_{xt} + \rho_x u u_x + \rho u_x^2 + \rho u u_{xx} + \rho_{xx} \theta + 2\rho_x \theta_x \right. \\ &\quad \left. + \rho \theta_{xx} + \mathbf{b} \cdot \mathbf{b}_{xx} + |\mathbf{b}_x|^2 \right\|_{L^2} \\ &\leq \|\rho_x\|_{L^2} \|u_t\|_{L^\infty} + \|\rho\|_{L^\infty} \|u_{xt}\|_{L^2} + \|\rho_x\|_{L^2} \|u\|_{L^\infty} \|u_x\|_{L^\infty} \\ &\quad + \|\rho\|_{L^\infty} \|u_x\|_{L^4}^2 + \|\rho\|_{L^\infty} \|u\|_{L^\infty} \|u_{xx}\|_{L^2} + \|\theta\|_{L^\infty} \|\rho_{xx}\|_{L^2} \\ &\quad + 2\|\rho_x\|_{L^\infty} \|\theta_x\|_{L^2} + \|\rho\|_{L^\infty} \|\theta_{xx}\|_{L^2} + \|b\|_{L^\infty} \|b_{xx}\|_{L^2} + \|b_x\|_{L^4}^2 \\ &\leq C \|u_t\|_{L^\infty} + C \|u_{xt}\|_{L^2} + C + C \|\rho_{xx}\|_{L^2} + C \|\rho_x\|_{L^\infty} + C \|\theta_{xx}\|_{L^2} \\ &\leq C \|u_{xt}\|_{L^2} + C + C \|\rho_{xx}\|_{L^2} + C \|\theta_{xx}\|_{L^2}. \end{aligned}$$

Inserting the above estimates into (2.41), and using Lemma 2.7 and the Gronwall inequality, we get

$$\int \rho_{xx}^2 dx + \iint_{Q_T} u_{xxx}^2 dx dt \leq C.$$

Since

$$\rho_{xt} = -(\rho u)_{xx},$$

it is easy to show that

$$\begin{aligned} \int \rho_{xt}^2 dx &\leq C \int (\rho^2 u_{xx}^2 + \rho_x^2 u_x^2 + \rho_{xx}^2 u^2) dx \\ &\leq C \int (u_{xx}^2 + \rho_{xx}^2) dx + C \|u_x\|_{L^\infty}^2 \int \rho_x^2 dx \leq C. \end{aligned}$$

Finally, noting

$$\rho_{tt} = -(\rho u)_{xt} = -(\rho_t u + \rho u_t)_x = -(\rho_{xt} u + \rho_t u_x + \rho_x u_t + \rho u_{xt}),$$

it holds that

$$\iint_{Q_T} \rho_{tt}^2 dx dt \leq C \|u\|_{L^\infty(Q_T)}^2 \iint_{Q_T} \rho_{xt}^2 dx dt + C \|\rho_t\|_{L^\infty(Q_T)}^2 \iint_{Q_T} u_x^2 dx dt$$

$$\begin{aligned}
& + C\|\rho_x\|_{L^\infty(Q_T)}^2 \iint_{Q_T} u_t^2 dxdt + C\|\rho\|_{L^\infty}^2 \iint_{Q_T} u_{xt}^2 dxdt \\
& \leq C \iint_{Q_T} (\rho_{xt}^2 + u_x^2 + u_t^2 + u_{xt}^2) dxdt \\
& \leq C + C \iint_{Q_T} u_{xt}^2 dxdt \leq C.
\end{aligned}$$

□

**Lemma 2.9.**

$$\int \rho \theta_t^2 dx + \iint_{Q_T} |(\kappa(\theta)\theta_x)_t|^2 dxdt \leq C. \quad (2.42)$$

*Proof.* Applying  $\partial_t$  to (1.5) gives

$$\begin{aligned}
& \rho \theta_{tt} + \rho u \theta_{xt} - (\kappa(\theta)\theta_x)_{xt} \\
& = 2(u_x u_{xt} + w_x w_{xt} + b_x b_{xt}) - p u_{xt} - p_t u_x - \rho_t \theta_t - \rho_t u \theta_x - \rho u_t \theta_x.
\end{aligned}$$

Multiplying the above equation by  $\kappa(\theta)\theta_t = \left(\int_0^\theta \kappa(\xi)d\xi\right)_t$ , integrating them over  $\Omega$ , and using (1.1),  $(\kappa\theta_t)_x = (\kappa\theta_x)_t$ , Lemmas 2.7 and 2.8, we infer that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \rho \kappa(\theta) \theta_t^2 dx + \int |(\kappa\theta_t)_x|^2 dx \\
& = \frac{1}{2} \int \rho \theta_t^2 (\kappa(\theta))_t dx + \frac{1}{2} \int \rho u \theta_t^2 (\kappa(\theta))_x dx \\
& \quad + 2 \int (u_x u_{xt} + \mathbf{w}_x \cdot \mathbf{w}_{xt} + \mathbf{b}_x \cdot \mathbf{b}_{xt}) \kappa(\theta) \theta_t dx \\
& \quad - \int (P u_{xt} + P_t u_x + \rho_t \theta_t + \rho_t u \theta_x + \rho u_t \theta_x) \kappa(\theta) \theta_t dx \\
& \leq \frac{1}{2} \|(\kappa(\theta))_t\|_{L^\infty} \int \rho \theta_t^2 dx + \frac{1}{2} \int \rho \theta_t^2 dx \|u\|_{L^\infty} \|(\kappa(\theta))_x\|_{L^\infty} \\
& \quad + 2(\|u_x\|_{L^2} \|u_{xt}\|_{L^2} + \|\mathbf{w}_x\|_{L^2} \|\mathbf{w}_{xt}\|_{L^2} + \|\mathbf{b}_x\|_{L^2} \|\mathbf{b}_{xt}\|_{L^2}) \|\kappa(\theta) \theta_t\|_{L^\infty} \\
& \quad + (\|P\|_{L^\infty} \|u_{xt}\|_{L^2} + \|\rho_t\|_{L^2} \|\theta\|_{L^\infty} \|u_x\|_{L^\infty} \\
& \quad + \|\sqrt{\rho} \theta_t\|_{L^2} \|\sqrt{\rho}\|_{L^\infty} \|u_x\|_{L^\infty}) \|\kappa(\theta) \theta_t\|_{L^\infty} \\
& \quad + \int (\rho u)_x \kappa \theta_t^2 dx + \|\rho_t\|_{L^2} \|u\|_{L^\infty} \|\theta_x\|_{L^2} \|\kappa(\theta) \theta_t\|_{L^\infty} \\
& \quad + \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho}\|_{L^\infty} \|\theta_x\|_{L^2} \|\kappa(\theta) \theta_t\|_{L^\infty} \\
& \leq C \|\kappa(\theta) \theta_t\|_{L^\infty} \int \rho \theta_t^2 dx + C \|\theta_{xx}\|_{L^2} \int \rho \theta_t^2 dx \\
& \quad + C(\|u_{xt}\|_{L^2} + \|\mathbf{w}_{xt}\|_{L^2} + \|\mathbf{b}_{xt}\|_{L^2}) \|\kappa(\theta) \theta_t\|_{L^\infty} \\
& \quad + C(1 + \|\sqrt{\rho} \theta_t\|_{L^2}) \|\kappa(\theta) \theta_t\|_{L^\infty} - \int \rho u (\kappa \theta_t^2)_x dx. \quad (2.43)
\end{aligned}$$

Noting that

$$\begin{aligned}
\|\kappa(\theta) \theta_t\|_{L^\infty} & \leq C \left( \int \rho \kappa(\theta) |\theta_t| dx + \|(\kappa(\theta) \theta_t)_x\|_{L^2} \right) \\
& \leq C \left( 1 + \int \rho \kappa(\theta) \theta_t^2 dx + \|(\kappa(\theta) \theta_t)_x\|_{L^2} \right), \quad (2.44)
\end{aligned}$$

and

$$\begin{aligned}
& - \int \rho u (\kappa(\theta) \theta_t^2)_x dx \\
& = - \int \rho u (\kappa(\theta) \theta_t)_x \theta_t dx - \int \rho u \kappa(\theta) \theta_t \theta_{xt} dx \\
& \leq \|u\|_{L^\infty} \|\sqrt{\rho} \theta_t\|_{L^2} \|\sqrt{\rho} (\kappa(\theta) \theta_t)_x\|_{L^2} \\
& \quad + \|u\|_{L^\infty} \|\sqrt{\rho} \theta_t\|_{L^2} \|\sqrt{\rho} [\kappa(\theta) \theta_t]_x - \kappa'(\theta) \theta_x \theta_t\|_{L^2} \\
& \leq C \|\sqrt{\rho} \theta_t\|_{L^2} \|(\kappa(\theta) \theta_t)_x\|_{L^2} + C \|\sqrt{\rho} \theta_t\|_{L^2}^2 \|\theta_x\|_{L^\infty} \\
& \leq C \|\sqrt{\rho} \theta_t\|_{L^2} \|(\kappa(\theta) \theta_t)_x\|_{L^2} + C \|\theta_{xx}\|_{L^2} \|\sqrt{\rho} \theta_t\|_{L^2}^2. \tag{2.45}
\end{aligned}$$

Inserting (2.44) and (2.45) into (2.43) and using the Gronwall inequality, we arrive at (2.42).  $\square$

**Lemma 2.10.**

$$\int \theta_{xx}^2 dx + \iint_{Q_T} \theta_{xxx}^2 dx \leq C. \tag{2.46}$$

*Proof.* It follows from (2.39) and (2.42) that

$$\int \theta_{xx}^2 dx \leq C. \tag{2.47}$$

It is easy to verify that

$$\int_0^T \|\theta_t\|_{L^\infty}^2 dt \leq \int_0^T \|\kappa(\theta) \theta_t\|_{L^\infty}^2 dt \leq C \int_0^T \|(\kappa(\theta) \theta_t)_x\|_{L^2}^2 dt + C \leq C. \tag{2.48}$$

Since

$$\kappa(\theta) \theta_{xt} = (\kappa(\theta) \theta_t)_x - \kappa'(\theta) \theta_t \theta_x,$$

by applying (2.48) and Cauchy inequality, we have

$$\begin{aligned}
\iint_{Q_T} \theta_{xt}^2 dx dt & \leq C \iint_{Q_T} \kappa^2(\theta) \theta_{xt}^2 dx dt \\
& \leq C \iint_{Q_T} |(\kappa(\theta) \theta_t)_x|^2 dx dt + C \iint_{Q_T} (\kappa'(\theta))^2 \theta_t^2 \theta_x^2 dx dt \\
& \leq C + C \int_0^T \|\theta_t\|_{L^\infty}^2 \|\theta_x\|_{L^2}^2 dt \\
& \leq C + C \int_0^T \|\theta_t\|_{L^\infty}^2 dt \leq C, \tag{2.49}
\end{aligned}$$

Applying the operator  $\partial_x$  to (1.5) gives

$$\begin{aligned}
\kappa(\theta) \theta_{xxx} & = -3\kappa'(\theta) \theta_x \theta_{xx} - \kappa''(\theta) \theta_x^3 - 2(u_x u_{xx} + \mathbf{w}_x \cdot \mathbf{w}_{xx} + \mathbf{b}_x \cdot \mathbf{b}_{xx}) \\
& \quad - \rho_x \theta_t - \rho \theta_{xt} - (\rho u \theta_x)_x - (\rho \theta u_x)_x,
\end{aligned}$$

whence

$$\begin{aligned}
\int \theta_{xxx}^2 dx & \leq \int \kappa^2(\theta) \theta_{xxx}^2 dx \leq C \int \theta_x^2 \theta_{xx}^2 dx + C \int \theta_x^6 dx \\
& \quad + C \int (u_x^2 u_{xx}^2 + |\mathbf{w}_x|^2 |\mathbf{w}_{xx}|^2 + |\mathbf{b}_x|^2 |\mathbf{b}_{xx}|^2) dx \\
& \quad + C \|\rho_x\|_{L^\infty}^2 \int \theta_t^2 dx + C \|\rho\|_{L^\infty}^2 \int \theta_{xt}^2 dx + C \|\rho u\|_{L^\infty}^2 \int \theta_{xx}^2 dx
\end{aligned}$$



$$\begin{aligned}
& + C\|\rho_x\|_{L^\infty}^2\|u\|_{L^\infty}^2\|\theta_x\|_{L^2}^2 + C\|\rho\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\|\theta_x\|_{L^2}^2 \\
& + C\|\rho_x\|_{L^\infty}^2\|\theta\|_{L^\infty}^2\|u_x\|_{L^2}^2 + C\|\rho\|_{L^\infty}^2\|\theta_x\|_{L^\infty}^2\|u_x\|_{L^2}^2 \\
& + C\|\rho\|_{L^\infty}^2\|\theta\|_{L^\infty}^2\|u_{xx}\|_{L^2}^2 \\
& \leq C + C \int \theta_t^2 dx + C \int \theta_{xt}^2 dx,
\end{aligned} \tag{2.50}$$

by Lemma 2.7, Lemma 2.8 and Lemma 2.9.

The estimates (2.48), (2.49) and (2.50) imply

$$\iint_{Q_T} \theta_{xxx}^2 dx dt \leq C.$$

□

By combining all the estimates obtained above, we get sufficient a priori estimates uniformly with  $\delta$  to take the limit  $\delta \rightarrow 0^+$  and extend the local strong solutions to be global one. Since the process is standard [6, 11], we omit them here for brevity. Hence the proof of Theorem 1.1 is completed.

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